

# MODULI STACKS OF VECTOR BUNDLES AND FROBENIUS MORPHISMS

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**ABSTRACT.** We describe the action of the different Frobenius morphisms on the cohomology ring of the moduli stack of algebraic vector bundles of fixed rank and determinant on an algebraic curve over a finite field in characteristic  $p$  and analyse special situations like vector bundles on the projective line and relations with infinite Grassmannians.

## INTRODUCTION

Suppose,  $F : X \rightarrow X$  denotes the geometric Frobenius endomorphism of a smooth, projective algebraic curve of genus  $g$  over the finite field  $\mathbb{F}_q$  of  $q = p^s$  elements of characteristic  $p > 0$ . It is well known and has been studied quite often, that the pullback operation on vector bundles on  $X$ , induced by  $F$ , does not necessarily respect the stability of the bundle. If the genus of the curve  $X$  is greater than one, not much is known about this situation.

In particular, the pullback operation  $F^*$  does not in general induce a morphism of the coarse moduli scheme of bundles in the sense of Narasimhan and Seshadri, but only a rational map. It is only recently, that Y. Laszlo and C. Pauly have been able to write down such a rational map explicitly in the special case of rank two vector bundles for  $p = 2$ ,  $g = 2$ , making use of the explicit knowledge of the coarse moduli scheme of (semi) stable bundles of rank 2 and degree 0 in the case  $g = 2$  ([15] and [16] and for more results [13]).

It is therefore natural to consider the action of  $F^*$  or  $\overline{F}^*$ , extending  $F$  to the algebraic closure  $\overline{\mathbb{F}}_q$  of  $\mathbb{F}_q$ , on the stack of bundles of rank  $n$  with trivial determinant as it is done in this paper.

Our contribution here consists of the observation (Prop. 2.4) that despite the fact that we do not know much about the action of  $F^*$ , it is possible to evaluate the action of  $F^*$  or  $\overline{F}^*$  on the cohomology ring of the moduli stack of vector bundles.

Of course, the next question is about the existence of a Lefschetz trace formula in this context, and we show by some easy examples that this can not work, at least not in a naive sense. In fact, already the case of the projective line  $X = \mathbb{P}^1$  with  $g = 0$  and for simplicity,  $n = 2$ , shows that

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there have to be modifications for such a trace formula taking care of fixed points at infinity of the stack of bundles. Formally, one can force convergence of the corresponding infinite sums (the stack of bundles has cohomology in infinitely many dimensions!) by twisting with a high enough power of the arithmetic Frobenius of the stack of bundles. Nevertheless this convergent expression is not directly related to the set of fixed points, as is shown already by the example above. It should be mentioned at this point that a Lefschetz trace formula for stacks works quite well for actions of the powers of the arithmetic Frobenius of the stack of bundles alone (without the pullback  $\overline{F}^*$ ) as is discussed in [3], [17].

The organisation of this paper is as follows: the first section contains some general material concerning stacks of vector bundles and describes the different Frobenius morphisms occurring.

Section two contains the cohomological computations and evaluates in particular the action of  $\overline{F}^*$  on the cohomology of the stack of bundles. Concerning the cohomology of the stack of bundles we have made use of work of Behrend [3] and in particular of the nice diploma thesis of J. Heinloth [12]. The procedure followed in their papers is parallel to the fundamental work of Atiyah and Bott [2], but whereas Atiyah and Bott work in a differential geometrical context the procedure here is in the language of algebraic geometry and stacks. The decisive point is the cohomology of the gauge group which has to be treated here in a different way using older results of Harder and Narasimhan on the Tamagawa number of the special linear group. For the convenience of the reader as part of the work [12] is unpublished, we will repeat here some of the arguments of [12] with hints to the literature and in particular to [5], where similar results are shown in a slightly different way.

The third section gives in part one the discussion of the example  $X = \mathbb{P}^1$  of the projective line and bundles of rank  $n = 2$ . The interesting situation is here, that thanks to Grothendieck's splitting theorem for vector bundles, the operation of  $\overline{F}^*$  on the stack is completely explicit. As is mentioned already, a trace formula in a naive sense does not hold. On the other hand we have the definite impression, that, if one could make sense of our approach in this case, this should generalize to the cases of arbitrary genus  $g$ , where not much is known about  $\overline{F}^*$ .

Part two of section three contains some further computations of the cohomology of moduli stacks of bundles and discusses the description of the stack of bundles as some kind of double quotient using the uniformisation theorem of Drinfeld and Simpson [7] (see also [21]). The ingredients of this description are the (infinite) affine Grassmannians from the theory of loop groups and an algebraic version of the gauge group. We indicate here a computation of the cohomology of both objects, evaluating along this a certain Leray spectral sequence. These considerations and the double coset description might be useful for extending our approach to a compactified situation.

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## 1. VECTOR BUNDLES ON CURVES AND FROBENIUS MORPHISMS

Let  $X$  denote a smooth, complete and irreducible algebraic curve of genus  $g$  over the finite field  $\mathbb{F}_q$  with  $q$  elements. For any scheme  $S$  over  $\mathbb{F}_q$ ,  $\tilde{\mathcal{L}}(S)$  denotes the following category: the objects of  $\tilde{\mathcal{L}}(S)$  are the rank  $n$  vector bundles over the scheme  $X \times_{\mathbb{F}_q} S$ , the morphisms of  $\tilde{\mathcal{L}}(S)$  are just isomorphisms between such vector bundles.

Additionally, we consider the category  $\mathcal{L}(S)$ , whose objects are pairs  $(E, \delta)$ , where  $E$  is a rank  $n$  vector bundle over  $X \times_{\mathbb{F}_q} S$  and  $\delta$  is an isomorphism of line bundles

$$\delta : \det(E) \xrightarrow{\sim} \mathcal{O}_{X \times_{\mathbb{F}_q} S}$$

of the determinant bundle of  $E$  with the trivial line bundle. We always identify vector bundles and their locally free module sheaves of sections. The morphisms  $u : (E, \delta) \rightarrow (E', \delta')$  in  $\mathcal{L}(S)$  are isomorphisms of vector bundles

$$u : E \xrightarrow{\sim} E',$$

inducing commutative diagrams

$$\begin{array}{ccc} \det(E) & \xrightarrow{\det(u)} & \det(E') \\ \delta \downarrow \sim & & \sim \downarrow \delta' \\ \mathcal{O}_{X \times_{\mathbb{F}_q} S} & \xrightarrow{\text{id}} & \mathcal{O}_{X \times_{\mathbb{F}_q} S} \end{array}$$

It is well known, that the functor

$$S/\mathbb{F}_q \mapsto \tilde{\mathcal{L}}(S)$$

on the category of schemes over  $\mathbb{F}_q$ , defines an algebraic stack in the sense of M. Artin ([17], théorème 4.6.2.1). Due to the fact that  $X$  is a curve over  $\mathbb{F}_q$ , it follows, that  $\mathcal{L}$  is a smooth algebraic stack. ([8], [12], 2.1.3, p.39).

Besides  $\tilde{\mathcal{L}}$ , we have also the stack  $\mathcal{L}$ , given by the functor  $S/\mathbb{F}_q \mapsto \mathcal{L}(S)$ , which is again smooth.

We will consider also the corresponding functors and stacks over  $\overline{\mathbb{F}}_q$ , denoted by  $\overline{\tilde{\mathcal{L}}}$  and  $\overline{\mathcal{L}}$  respectively. From the definitions it follows immediately that there is a canonical isomorphism of stacks

$$\begin{aligned}\tilde{\mathcal{L}} \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q &\xrightarrow{\sim} \overline{\tilde{\mathcal{L}}}, \\ \mathcal{L} \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q &\xrightarrow{\sim} \overline{\mathcal{L}}.\end{aligned}$$

On  $X \times_{\mathbb{F}_q} \tilde{\mathcal{L}}$  we have the universal vector bundle  $E^{\text{univ}}$ , such that for any scheme  $S$  over  $\mathbb{F}_q$  a morphism  $\varphi : S \rightarrow \tilde{\mathcal{L}}$  induces the vector bundle  $(\text{id}_X \times \varphi)^*(E^{\text{univ}})$  on  $X \times_{\mathbb{F}_q} S$ , which is exactly the element of  $\tilde{\mathcal{L}}(S)$  given by  $\varphi$ .

In the same way, on  $X \times_{\mathbb{F}_q} \mathcal{L}$ , we have a universal pair  $(E^{\text{univ}}, \delta_{\text{univ}})$ , where  $\delta_{\text{univ}} : \det(E^{\text{univ}}) \xrightarrow{\sim} \mathcal{O}_{X \times_{\mathbb{F}_q} \mathcal{L}}$ . Similar considerations are true over the algebraic closure  $\overline{\mathbb{F}}_q$ .

Following [2], [8], [12], we will consider various open substacks of  $\tilde{\mathcal{L}}$  resp.  $\mathcal{L}$  and similarly for  $\overline{\tilde{\mathcal{L}}}$  resp.  $\overline{\mathcal{L}}$ .

For an arbitrary vector bundle  $E$  over an algebraic curve  $X$  (over a field),  $\mu(E)$  denotes the quotient

$$\mu(E) := \frac{\deg(E)}{\text{rk}(E)},$$

where  $\deg(E)$  is the degree of the bundle  $E$ ,  $\text{rk}(E)$  its rank.

**Definition 1.1.** *A vector bundle  $E$  is stable (resp. semistable) if for all proper subsheaves (resp. all subsheaves)  $E'$  of  $E$  the inequality  $\mu(E') < \mu(E)$  (resp.  $\mu(E') \leq \mu(E)$ ) holds.*

As is well known, any vector bundle  $E$  on a curve  $X$  has a canonical filtration, the *Harder-Narasimhan filtration*:

$$E_0 = (0) \subset E_1 \subset \dots \subset E_r = E,$$

such that the following properties hold:

- i) The successive quotients  $E_i/E_{i-1}$  are all semistable.
- ii) Given  $E_i, E_{i+1}$  is a maximal subbundle of  $E$  with the additional property that

$$\mu(E_{i+1}/E_i) \geq \mu(E')$$

holds for all subbundles  $E' \subset E/E_i$ . Given  $E_i$ , this determines  $E_{i+1}$  uniquely.

- iii) Dually, given  $E_i$ , then  $E_{i-1}$  is the smallest subbundle of  $E_i$  with the property, that

$$\mu(E_i/E_{i-1}) \leq \mu(E'')$$

for all quotient bundles  $E_i/E''$ .

The Harder-Narasimhan filtration is uniquely determined by the above properties. We associate now with this filtration the polygonal function with  $\text{rk}(E) = n$

$$p_E : [0, \text{rk}(E)] \rightarrow \mathbb{R},$$

given as follows:  $p_E(\text{rk}(E_i)) = \deg(E_i)$  for  $i = 0, \dots, r$ . At all other values  $x \in [0, \text{rk}(E)]$ , the value  $p_E(x)$  is obtained by linear interpolation using the values defined before. Obviously  $p_E$  is a concave function, that is, slopes are decreasing with growing  $x$ .

For any piecewise linear function  $p : [0, \text{rk}(E)] \rightarrow \mathbb{R}$ , satisfying  $p(\text{rk}(E)) = \deg(E) = 0$  (we assume, that  $\det(E) \cong \mathcal{O}_X$ ), we define substacks  $\mathcal{L}_{\leq p}$  resp.  $\mathcal{L}_{< p}$ .

**Definition 1.2.** For any scheme  $S$  over  $\mathbb{F}_q$ ,  $\mathcal{L}_{\leq p}(S)$  is the subcategory of  $\mathcal{L}(S)$  of pairs  $(E, \delta)$ ,  $E$  a bundle over  $X \times_{\mathbb{F}_q} S$  of rank  $n$ ,  $\delta : \det(E) \xrightarrow{\sim} \mathcal{O}_{X \times_{\mathbb{F}_q} S}$  a trivialisation of the determinant bundle, such that for all closed points  $s \in S$ , the Harder-Narasimhan polygon  $p_{E_s}$  of  $E \times_{\mathbb{F}_q} s$  satisfies  $p_{E_s} \leq p$ . Similarly,  $\mathcal{L}_{< p}(S)$  denotes the subcategory of pairs  $(E, \delta)$ , such that  $p_{E_s} < p$  holds for all closed points  $s \in S$ .

**Proposition 1.3.** The stacks  $\mathcal{L}_{< p} \hookrightarrow \mathcal{L}$  resp.  $\mathcal{L}_{\leq p} \hookrightarrow \mathcal{L}$  are open substacks of  $\mathcal{L}$ . Similarly  $\mathcal{L}_{< p} \hookrightarrow \mathcal{L}_{\leq p}$  is an open substack.  $\mathcal{L}_p$  denotes the reduced closed substack  $(\mathcal{L}_{\leq p} \setminus \mathcal{L}_{< p})$ .

**Proof.** The proposition follows by using the semicontinuity theorem. For the details see for example [12], 2.1.10.  $\square$

**Remark.** Similarly we have the open substacks  $\overline{\mathcal{L}}_{\leq p} = \mathcal{L}_{\leq p} \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$  and  $\overline{\mathcal{L}}_{< p} = \mathcal{L}_{< p} \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$  of  $\overline{\mathcal{L}}$  as well as the closed substack  $\overline{\mathcal{L}}_p = \mathcal{L}_p \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$  of  $\overline{\mathcal{L}}$ .

We recall for completeness

**Definition 1.4.**  $(E^{\text{univ}}, \delta^{\text{univ}})$  denotes the universal rank  $n$  vector bundle over  $X \times_{\mathbb{F}_q} \mathcal{L}$  with trivial determinant  $\delta^{\text{univ}} : \det(E) \xrightarrow{\sim} \mathcal{O}_{X \times_{\mathbb{F}_q} \mathcal{L}}$ .

We consider the following Frobenius morphisms in this context. First we have the geometric Frobenius endomorphism

$$F_X : (X, \mathcal{O}_X) \rightarrow (X, \mathcal{O}_X),$$

$$F_X = \text{id (on } X), F_X^*(f) = f^q \text{ (on sections of } \mathcal{O}_X).$$

Let  $\overline{F} := F_X \times_{\mathbb{F}_q} \text{id}_{\overline{\mathbb{F}}_q}$  denote the extension of  $F_X$  to an endomorphism of  $X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$  over  $\overline{\mathbb{F}}_q$ .

If  $S/\overline{\mathbb{F}}_q$  denotes a scheme over  $\overline{\mathbb{F}}_q$ , one has the pullback operation

$$\begin{aligned} \overline{\mathcal{L}}(S) &\rightarrow \overline{\mathcal{L}}(S) \\ (E, \delta) &\mapsto (\overline{F}^*(E), \overline{F}^*(\delta)). \end{aligned}$$

This induces an endomorphism of the stack  $\overline{\mathcal{L}}$ ,

$$\varphi : \overline{\mathcal{L}} \rightarrow \overline{\mathcal{L}}.$$

**Remark.** As already outlined in the introduction, the nature of  $\varphi$  is very mysterious. In particular it does not respect the open substacks  $\overline{\mathcal{L}}_{\leq p} \hookrightarrow \overline{\mathcal{L}}$  or  $\overline{\mathcal{L}}_{< p} \hookrightarrow \overline{\mathcal{L}}$ .

Besides  $\varphi$  we have also the geometric Frobenius morphism

$$F_{\mathcal{L}} : (\mathcal{L}, \mathcal{O}_{\mathcal{L}}) \rightarrow (\mathcal{L}, \mathcal{O}_{\mathcal{L}})$$

and its extension  $F_{\mathcal{L}} \times_{\mathbb{F}_q} \text{id}_{\overline{\mathbb{F}}_q}$  as an endomorphism of  $\overline{\mathcal{L}}$  over  $\overline{\mathbb{F}}_q$ . Of course this endomorphism respects the open substacks  $\mathcal{L}_{\leq p} \hookrightarrow \mathcal{L}$ ,  $\mathcal{L}_{< p} \hookrightarrow \mathcal{L}$  and their extensions over  $\overline{\mathbb{F}}_q$ .

Actually we prefer to work in the next chapters with the arithmetic Frobenius morphism on the smooth-étale site  $(\mathcal{L} \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q)_{\text{Sm-ét}}$ , given as  $\psi := \text{id}_{\mathcal{L}} \times \text{Frob}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$  which acts as an inverse to the action of  $F_{\mathcal{L}} \times_{\mathbb{F}_q} \text{id}_{\overline{\mathbb{F}}_q}$  on the smooth-étale site  $(\mathcal{L} \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q)_{\text{Sm-ét}}$ .

**Proposition 1.5.** *There is a canonical isomorphism*

$$(\overline{F}_X \times \text{id}_{\overline{\mathcal{L}}})^*(E^{\text{univ}}) \cong (\text{id}_{\overline{X}} \times \varphi)^*(E^{\text{univ}}).$$

**Proof.** This is immediate. For a stack  $T/\overline{\mathbb{F}}_q$  a rank  $n$  vector bundle  $E$  on  $\overline{X} \times T$  together with an isomorphism

$$\det(E) \xrightarrow{\sim} \mathcal{O}_{\overline{X} \times_{\mathbb{F}_q} T}$$

is given by a morphism  $u : T \rightarrow \overline{\mathcal{L}}$ , such that  $E \cong (\text{id}_{\overline{X}} \times u)^*(E^{\text{univ}})$  and similarly for the determinant.

Applying this to the vector bundle  $(\overline{F}_X \times_{\overline{\mathbb{F}}_q} \text{id}_{\overline{\mathcal{L}}})^*(E^{\text{univ}})$ , which defines  $\varphi : \overline{\mathcal{L}} \rightarrow \overline{\mathcal{L}}$ , and the proposition follows.  $\square$

## 2. COHOMOLOGY OF THE MODULI STACKS OF VECTOR BUNDLES

In this section we describe the cohomology of the moduli stack of vector bundles. We collect the results from various different treatments in the literature. (see [12], [3], [5] and [2]).

Let  $X$  be an algebraic curve over the field  $\mathbb{F}_q$ . We have the following well known description of the  $l$ -adic cohomology ring of the curve  $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$  over the algebraic closure  $\overline{\mathbb{F}}_q$ , where  $\overline{\mathbb{Q}}_l$  denotes the algebraic closure of  $\mathbb{Q}_l$ , namely

$$\begin{aligned} H^0(\overline{X}; \overline{\mathbb{Q}}_l) &= \overline{\mathbb{Q}}_l \cdot 1 \\ H^1(\overline{X}; \overline{\mathbb{Q}}_l) &= \bigoplus_{i=1}^{2g} \overline{\mathbb{Q}}_l \cdot \alpha_i \\ H^2(\overline{X}; \overline{\mathbb{Q}}_l) &= \overline{\mathbb{Q}}_l[\overline{X}], \end{aligned}$$

where  $[\overline{X}]$  denotes the orientation class of  $\overline{X}/\overline{\mathbb{F}}_q$  and where we can assume, that the  $\{\alpha_i : i = 1, \dots, 2g\}$  are eigenclasses under the action of the Frobenius morphism. The Frobenius morphism acts semisimple on the étale cohomology of a curve (see [18], p.203).

More precisely, for  $\overline{F} = F_X \times_{\mathbb{F}_q} \text{id}_{\overline{\mathbb{F}}_q}$  as before, we have explicitly that

$$\begin{aligned} \overline{F}^*(1) &= 1 \\ \overline{F}^*(\alpha_i) &= \lambda_i \alpha_i \quad (i = 1, \dots, 2g), \end{aligned}$$

where  $\lambda_i \in \overline{\mathbb{Q}}_l$  is algebraic, such that  $|\lambda_i| = q^{\frac{1}{2}}$  for any possible embedding of  $\lambda_i$  into the complex numbers  $\mathbb{C}$ . Finally  $\overline{F}^*([\overline{X}]) = q \cdot [\overline{X}]$  for the orientation class  $[\overline{X}]$ .

The definition of the  $l$ -adic cohomology of an algebraic stack  $\mathcal{X}$  can be found in [17], chapter 12. In [3], [12] the authors work with the smooth topology on algebraic stacks. We will follow here the treatment of [17]. For an algebraic stack  $\mathcal{X}$  over a scheme  $S$ ,  $\mathcal{X}/S$ , we consider the smooth-étale site  $\text{Sm-et}(\mathcal{X}/S)$  ([17], Def. 12.1.) The underlying category of  $\text{Sm-et}(\mathcal{X}/S)$  has as objects the smooth 1-morphisms of  $S$ -algebraic stacks  $u : U \rightarrow \mathcal{X}$ , where  $U$  is an algebraic space over  $S$ . A morphism is a pair  $(\varphi, \alpha) : (U, u) \rightarrow (V, v)$ , where  $\varphi : U \rightarrow V$  is a morphism of  $S$ -algebraic spaces and  $\alpha$  is a 2-isomorphism  $\alpha : u \xrightarrow{\sim} (v \circ \varphi)$ . The topology on this category is generated by the pretopology of families of morphisms

$$((\varphi_i, \alpha_i) : (U_i, u_i) \rightarrow (U, u))_{i \in I},$$

where the 1-morphism  $\coprod_{i \in I} \varphi : \coprod_{i \in I} U_i \rightarrow U$  is étale and surjective.

The  $l$ -adic cohomology of a stack  $\mathcal{X}$  is defined with respect to this site  $\mathcal{X}_{\text{Sm-et}}$  as

$$H^\bullet(\mathcal{X}; \mathbb{Q}_l) = (\varprojlim_{\overline{n}} H^\bullet(\mathcal{X}; \mathbb{Z}/l^n \mathbb{Z})) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

Basically, the method is to take a covering  $X \rightarrow \mathcal{X}$  (over  $\overline{\mathbb{F}}_q$ ) of the stack  $\mathcal{X}$  by a scheme  $X$  and to consider the cohomology of the associated simplicial scheme

$$[X/\mathcal{X}] := [X \rightrightarrows X \times_{\mathcal{X}} X \xrightarrow[\rightrightarrows]{\rightrightarrows} X \times_{\mathcal{X}} X \times_{\mathcal{X}} X \dots]$$

and an appropriate smooth-étale site over it. For details, see chapters 12 and 18 of the book [17] or [21].

We describe now in detail the steps of the computation of the cohomology  $H^\bullet(\overline{\mathcal{L}}; \mathbb{Q}_l)$  using the treatment of [5] and comparing it with [3] and [12]. The definition of  $l$ -adic cohomology above causes no problems for the algebraic stack of bundles  $\overline{\mathcal{L}}$ , as  $\overline{\mathcal{L}}$  is the inductive limit of the open substacks  $\overline{\mathcal{L}}_{\leq p}$  and the cohomology of this inductive system over the Harder-Narasimhan polygons  $p$  is constant for large  $p$  as will be discussed below.

1) As described in section 1, Definition 1.3. and Proposition 1.4., we have upon fixing a Harder-Narasimhan polygon  $p$ , the open substacks,  $\overline{\mathcal{L}}_{<p} \hookrightarrow \overline{\mathcal{L}}$  as well as  $\overline{\mathcal{L}}_{\leq p} \hookrightarrow \overline{\mathcal{L}}$ . Similarly  $\overline{\mathcal{L}}_{<p} \hookrightarrow \overline{\mathcal{L}}_{\leq p}$  is an open substack,  $\overline{\mathcal{L}}_p$  denotes the reduced closed substack  $(\overline{\mathcal{L}}_{\leq p} \setminus \overline{\mathcal{L}}_{<p})$ .

In particular, because  $X$  is one-dimensional,  $\overline{\mathcal{L}}_{\leq p}, \overline{\mathcal{L}}_{<p}, \overline{\mathcal{L}}_p$  are smooth algebraic stacks and  $\overline{\mathcal{L}}_p \hookrightarrow \overline{\mathcal{L}}_{\leq p}$  is a smooth pair. This can be found in a similar way also in [5], section 5.

2) For the smooth pair  $i : \overline{\mathcal{L}}_p \hookrightarrow \overline{\mathcal{L}}_{\leq p}$  one has the tangent stacks  $T(\overline{\mathcal{L}}_p)$  and  $T(\overline{\mathcal{L}}_{\leq p})$ , ([17], 17.11 – 17.15 and 17.16 – 17.17), as well as the normal bundle

$$N(\overline{\mathcal{L}}_{\leq p}/\overline{\mathcal{L}}_p) = [i^*T(\overline{\mathcal{L}}_{\leq p})/T(\overline{\mathcal{L}}_p)].$$

The normal bundle has the following explicit description in terms of the universal bundle  $(E^{\text{univ}}, \delta^{\text{univ}})$  (see Definition 1.5).

Upon restricting  $(E^{\text{univ}}, \delta^{\text{univ}})|_{\overline{\mathcal{L}}_p}$ , one has the canonical sequence of endomorphism bundles

$$0 \rightarrow \text{End}_{\text{filt}}^{(0)}(E^{\text{univ}}|\overline{\mathcal{L}}_p) \rightarrow \text{End}^{(0)}(E^{\text{univ}}|\overline{\mathcal{L}}_p) \rightarrow \widetilde{\text{End}}^{(0)}(E^{\text{univ}}|\overline{\mathcal{L}}_p) \rightarrow 0,$$

where “(0)” denotes the “trace=0” endomorphism bundles and “filt” specifies the endomorphisms respecting the Harder-Narasimhan filtration (fibrewise). It is easy to see, that there is an isomorphism of bundles

$$N(\overline{\mathcal{L}}_{\leq p}/\overline{\mathcal{L}}_p) = R^1\text{pr}_*(\widetilde{\text{End}}^{(0)}(E^{\text{univ}}|\overline{\mathcal{L}}_p))$$

(see [12], 1.3, p. 37 and [5], Prop. 5.2, in particular 5.2.(3)).

3) We have a morphism of stacks

$$f : \overline{\mathcal{L}}_p \rightarrow \left( \prod_{i=1}^l \overline{\mathcal{L}}(X; n_i; d_i) \right)^{(0)},$$

which associates with a vector bundle of a fixed Harder-Narasimhan filtration of type  $p$  the semistable subquotients occurring in the filtration. “(0)” here denotes the bundles equipped with a trivialisation of the total determinant. This morphism of algebraic stacks is not representable. Nevertheless it is not difficult to conclude, using the associated simplicial schemes, that  $f$  is cohomologically acyclic inducing in particular an isomorphism on cohomology (with constant coefficients), (see [12], 2.1.1. and [5], Prop. 7.1, 7.2)).

4) For the smooth pair  $(\overline{\mathcal{L}}_{\leq p}, \overline{\mathcal{L}}_p)$  one has a Gysin sequence

$$\begin{aligned} \dots &\rightarrow H^{\bullet-2c}(\overline{\mathcal{L}}_p; \mathbb{Z}/l^n\mathbb{Z}(c)) \xrightarrow{i_0} H^{\bullet}(\overline{\mathcal{L}}_{\leq p}; \mathbb{Z}/l^n\mathbb{Z}) \\ &\rightarrow H^{\bullet}(\overline{\mathcal{L}}_{<p}; \mathbb{Z}/l^n\mathbb{Z}) \longrightarrow H^{\bullet+1-2c}(\overline{\mathcal{L}}_p; \mathbb{Z}/l^n\mathbb{Z}(c)) \rightarrow \dots \end{aligned}$$

where  $c$  is the codimension of the stack  $\overline{\mathcal{L}}_p$  in the stack  $\overline{\mathcal{L}}_{\leq p}$ .

**Proof.** This is [3], Prop. 2.1.2 and Corollary 2.1.3.



5) The composition  $i^*i_*$  is given as the cup-product

$$i^*i_*(x) = c_r(N^*) \cup x,$$

where  $N = N(\overline{\mathcal{L}}_{\leq p}/\overline{\mathcal{L}}_p)$  is the normal bundle computed above,  $N^*$  the dual bundle,  $c_r(N^*)$  is the top Chern class of the bundle  $N^*$ .

**Proof.** For schemes this is [20], Exposé VII, Théorème 4.1., p. 299. The case of stacks can be treated in the same way.

6) The cup product with  $c_r(N^*)$  in the situation above is always injective. As a consequence the Gysin sequence above for the pair  $(\overline{\mathcal{L}}_{\leq p}; \overline{\mathcal{L}}_p)$  splits completely into short exact sequences

$$0 \rightarrow H^{\bullet-2c}(\overline{\mathcal{L}}_p) \rightarrow H^{\bullet}(\overline{\mathcal{L}}_{\leq p}) \rightarrow H^{\bullet}(\overline{\mathcal{L}}_{< p}) \rightarrow 0.$$

The  $\mathbb{Z}/l^n\mathbb{Z}$ -modules  $H^{\bullet}(\overline{\mathcal{L}}_{\leq p}; \mathbb{Z}/l^n\mathbb{Z})$  are free of finite rank.

**Proof.** Using induction over  $n$  and the order of the Harder-Narasimhan polygon, it suffices to treat the case  $n = 1$ . Using 3), one is reduced to the corresponding statement for  $\overline{\mathcal{L}}(X; n; d)_{ss}$  and products of these resp. to the “0”-versions of these. Using the usual geometric invariant theory description of the moduli space of vector bundles, such a product can be written as a quotient stack  $[G \backslash \prod_{i=1}^l Gr_m(n_i; d_i)]$  of a product of Grassmannians by the action of a connected algebraic group  $G$ . By ([3], Theorem 1.4.3), the spectral sequence describing this quotient degenerates and in particular the cohomology of the stack injects into the cohomology of the corresponding quotient stack  $[T \backslash \prod_{i=1}^l Gr_m(n_i; d_i)]$ , on which it is easy to check the injectivity of the cup product above. A different proof can be derived also from [5], in particular using Prop. 4.2, Prop. 7.1.  $\square$

Using a Mittag-Leffler type argument one obtains now

$$H^{\bullet}(\overline{\mathcal{L}}; \mathbb{Q}_l) = \varprojlim_{(p)} H^{\bullet}(\overline{\mathcal{L}}_{\leq p}; \mathbb{Q}_l)$$

and it follows

**Proposition 2.1.** *The Gysin sequence for the pair  $(\overline{\mathcal{L}}_{\leq p}, \overline{\mathcal{L}}_p)$  splits into the short exact sequences*

$$0 \rightarrow H^{*-2r}(\overline{\mathcal{L}}_p; \overline{\mathbb{Q}}_l(r)) \xrightarrow{i_*} H^*(\overline{\mathcal{L}}_{\leq p}; \overline{\mathbb{Q}}_l) \rightarrow H^*(\overline{\mathcal{L}}_{< p}; \overline{\mathbb{Q}}_l) \rightarrow 0,$$

where  $r$  is the codimension of the stack  $\overline{\mathcal{L}}_p$  in  $\overline{\mathcal{L}}_{\leq p}$ .

7) Now we have the following result concerning the Poincaré series of the cohomology ring of the moduli stack  $\overline{\mathcal{L}}$

**Proposition 2.2.** *The Poincaré series of the cohomology ring  $H^*(\overline{\mathcal{L}}; \overline{\mathbb{Q}}_l)$  is given as*

$$P(\overline{\mathcal{L}}; t) = \frac{\prod_{i=1}^n (1 + t^{2i-1})^{2g}}{\prod_{i=2}^n (1 + t^{2i}) \prod_{i=2}^n (1 - t^{2i-2})}.$$

**Proof.** This follows now directly from Proposition 2.1 using [5], Prop. 4.2, Prop. 8.1, and Prop. 10.1.. An alternative, more arithmetic proof is given in [12], Satz 2.2.6, but with the slight modification, that we are considering the case with trivial determinant. This proof uses again Proposition 2.1., the Lefschetz trace formula (see [3]) and the Tamagawa number computations from [11].  $\square$

8) The cohomology of  $\overline{\mathcal{L}}$  can be obtained now as follows: we consider on  $\overline{X} \times \overline{\mathcal{L}}$  the universal bundle  $E^{\text{univ}}$  and its Chern classes

$$c_i(E^{\text{univ}}) \in H^{2i}(\overline{X} \times \overline{\mathcal{L}}; \overline{\mathbb{Q}}_l) \quad (i \geq 2).$$

**Remark.** The first Chern class  $c_1(E^{\text{univ}})$  vanishes because we have by definition the trivialisation  $\delta^{\text{univ}} : \det(E^{\text{univ}}) \xrightarrow{\sim} \mathcal{O}_{\overline{X} \times \overline{\mathcal{L}}}$ .

We apply now Künneth's theorem, which holds also in the situation here, because we can view  $\overline{\mathcal{L}}$  as a simplicial scheme using a representation of  $\overline{\mathcal{L}}$  as mentioned above and therefore also the product  $\overline{X} \times \overline{\mathcal{L}}$  as a simplicial scheme. Then we can apply Künneth's theorem degreewise for the simplicial scheme representing  $\overline{X} \times \overline{\mathcal{L}}$  to obtain it for  $\overline{X} \times \overline{\mathcal{L}}$  by standard simplicial techniques. Therefore we can decompose the Chern classes above as follows:

$$c_i(E^{\text{univ}}) = 1 \otimes c_i + \sum_{j=1}^{2g} \alpha_j \otimes a_i^{(j)} + [X] \otimes b_{i-1}.$$

Here we have, for  $i \geq 2$ ,  $c_i \in H^{2i}(\overline{\mathcal{L}}; \overline{\mathbb{Q}}_l)$ ,  $a_i^{(j)} \in H^{2i-1}(\overline{\mathcal{L}}; \overline{\mathbb{Q}}_l)$  for  $j = 1, \dots, 2g$  and  $b_{i-1} \in H^{2(i-1)}(\overline{\mathcal{L}}; \overline{\mathbb{Q}}_l)$ .

**Proposition 2.3.** *There is an isomorphism of graded  $\overline{\mathbb{Q}}_l$ -algebras*

$$H^*(\overline{\mathcal{L}}; \overline{\mathbb{Q}}_l) \cong \overline{\mathbb{Q}}_l[c_2, \dots, c_n; b_1, \dots, b_{n-1}] \otimes \bigotimes_{i=1}^n \Lambda[a_i^{(1)}, \dots, a_i^{(2g)}],$$

where  $\mathbb{Q}_l[c_2, \dots, c_n; b_1, \dots, b_{n-1}]$  is the (graded) polynomial algebra in the generators  $c_2, \dots, c_n, b_1, \dots, b_{n-1}$  and  $\Lambda[a_i^{(1)}, \dots, a_i^{(2g)}]$  is an exterior algebra with generators  $a_i^{(1)}, \dots, a_i^{(2g)}$  in degree  $(2i-1)$ .

**Proof.** The proof can be found in [12], Satz 2.2.8 or [5], again of course with the slight modification, that we are considering the case of trivial determinant. The strategy of the proof can be outlined as follows: first, upon

restriction to the closed substack of  $\bar{\mathcal{L}}$ , consisting of vector bundles, which are direct sums of line bundles, it follows that the canonical map

$$\bar{\mathbb{Q}}_l[c_1, \dots, c_n; b_1, \dots, b_{n-1}] \otimes \bigotimes_{i=1}^n \Lambda[a_i^{(1)}, \dots, a_i^{(2g)}] \rightarrow H^*(\bar{\mathcal{L}}; \bar{\mathbb{Q}}_l),$$

defined by sending the generators  $c_i, b_i, a_i^{(j)}$  to the corresponding cohomology classes in  $H^*(\bar{\mathcal{L}}; \bar{\mathbb{Q}}_l)$ , is injective. The surjectivity follows then from comparing the Poincaré polynomials of the two graded algebras using Proposition 2.2.  $\square$

We will study now the action induced by the morphism  $\varphi : \bar{\mathcal{L}} \rightarrow \bar{\mathcal{L}}$  of stacks on cohomology. By Proposition 1.6. we have the following isomorphism of vector bundles on  $\bar{X} \times \bar{\mathcal{L}}$ ,

$$(\bar{F}_X \times \text{id}_{\bar{\mathcal{L}}})(E^{\text{univ}}) \cong (\text{id}_{\bar{X}} \times \varphi)^*(E^{\text{univ}})$$

This induces the equalities of Chern classes ( $i \geq 2$ )

$$c_i((\bar{F}_X \times \text{id}_{\bar{\mathcal{L}}})^*(E^{\text{univ}})) = c_i((\text{id}_{\bar{X}} \times \varphi)^*(E^{\text{univ}}))$$

Using the functoriality of Chern classes, we obtain therefore

$$(F_{\bar{X}} \times \text{id}_{\bar{\mathcal{L}}})^*(c_i(E^{\text{univ}})) = (\text{id}_{\bar{X}} \times \varphi)^*(c_i(E^{\text{univ}}))$$

From above we have the formula

$$c_i(E^{\text{univ}}) = 1 \otimes c_i + \sum_{j=1}^{2g} \alpha_j \otimes a_i^{(j)} + [X] \otimes b_{i-1}$$

for  $i \geq 2$ . Therefore we obtain the equality

$$\begin{aligned} & 1 \otimes c_i + \sum_{j=1}^{2g} \lambda_j \alpha_j \otimes a_i^{(j)} + q[X] \otimes b_{i-1} \\ &= 1 \otimes \varphi^*(c_i) + \sum_{j=1}^{2g} \alpha_j \otimes \varphi^*(a_i^{(j)}) + [X] \otimes \varphi^*(b_{i-1}) \end{aligned}$$

This implies the following equalities

$$\begin{aligned} \varphi^*(c_i) &= c_i & (i \geq 1) \\ \varphi^*(a_i^{(j)}) &= \lambda_j a_i^{(j)} & (i \geq 1, j = 1, \dots, 2g) \\ \varphi^*(b_i) &= q b_i & (i \geq 1) \end{aligned}$$

and we have proved:

**Proposition 2.4.** *The morphism  $\varphi : \bar{\mathcal{L}} \rightarrow \bar{\mathcal{L}}$  of the stack of vector bundles of rank  $n$  on  $\bar{X}$ , given by the pullback operation induced by  $\bar{F}_X : \bar{X} \rightarrow \bar{X}$ , acts on the generating cohomology classes  $c_i, a_i^{(j)}, b_i$  of  $H^*(\bar{\mathcal{L}}; \bar{\mathbb{Q}}_l)$  as:*

$$\begin{aligned} \varphi^*(c_i) &= c_i & (i \geq 1) \\ \varphi^*(a_i^{(j)}) &= \lambda_j a_i^{(j)} & (i \geq 1, j = 1, \dots, 2g) \\ \varphi^*(b_i) &= q b_i & (i \geq 1) \end{aligned}$$

**Remark.** As mentioned earlier, it is somewhat surprising that one can determine the action of  $\varphi$  on the cohomology of  $\mathcal{L}$  in such an explicit way, because otherwise the nature of  $\varphi$  is mysterious.

We will use now similar considerations to study the action of the genuine Frobenius endomorphism  $\overline{F}_{\mathcal{L}} = F_{\mathcal{L}} \times \text{id}_{\overline{\mathbb{F}}_q/\mathbb{F}_q}$  on the cohomology  $H^*(\tilde{\mathcal{L}}; \overline{\mathbb{Q}}_l)$ . To proceed further, we consider the classifying stacks of all rank  $n$  vector bundles  $BGL(n)/\mathbb{F}_q$  and  $BGL(n)/\mathbb{F}_q \times \overline{\mathbb{F}}_q$ . As is well known, (see [3], Th. 2.3.2.) one has

$$H^*(BGL(n) \times \overline{\mathbb{F}}_q; \overline{\mathbb{Q}}_l) \simeq \overline{\mathbb{Q}}_l[c_1, \dots, c_n]$$

and the geometric Frobenius  $\overline{F}_{BGL(n)/\mathbb{F}_q}$  acts as

$$(\overline{F}_{BGL(n)/\mathbb{F}_q})^*(c_i) = q^i c_i \text{ for } i \geq 1.$$

Using this we can obtain more information about the Frobenius action on the cohomology  $H^*(\tilde{\mathcal{L}}; \overline{\mathbb{Q}}_l)$ . We have

$$(\overline{F}_X \times \text{id}_{\tilde{\mathcal{L}}})^*(\text{id}_{\tilde{\mathcal{L}}} \times \overline{F}_{\mathcal{L}})^*(c_i(E^{\text{univ}})) \cong (\overline{F}_{X \times \mathcal{L}})^*(c_i(E^{\text{univ}})) \cong (\overline{F}_{X \times \mathcal{L}})^* c_i(u^* \tilde{E}^{\text{univ}}),$$

where  $\tilde{E}^{\text{univ}}$  denotes the universal vector bundle on the classifying stack  $BGL(n)/\mathbb{F}_q$  and  $u : X \times \mathcal{L} \rightarrow BGL(n)$  denotes the classifying morphism for  $E^{\text{univ}}$  on  $(X \times \mathcal{L})$ , such that

$$u^*(\tilde{E}^{\text{univ}}) \cong E^{\text{univ}}$$

with a canonical isomorphism. Furthermore we have the commutative diagram

$$\begin{array}{ccc} X \times \mathcal{L} & \xrightarrow{u} & BGL(n)/\mathbb{F}_q \\ F_{X \times \mathcal{L}} \downarrow & & \downarrow F_{BGL(n)/\mathbb{F}_q} \\ X \times \mathcal{L} & \xrightarrow{u} & BGL(n)/\mathbb{F}_q \end{array}$$

and similarly after extension from  $\mathbb{F}_q$  to  $\overline{\mathbb{F}}_q$ . Therefore we conclude

$$\begin{aligned} (\overline{F}_{X \times \mathcal{L}})^* c_i(\tilde{u}^*(\tilde{E}^{\text{univ}})) &= (\overline{F}_{X \times \mathcal{L}})^*(\tilde{u})^* c_i(\tilde{E}^{\text{univ}}) \\ &= (\tilde{u})^*(\overline{F}_{BGL(n)/\mathbb{F}_q})^*(c_i(\tilde{E}^{\text{univ}})) \\ &= q^i (\tilde{u})^*(c_i(\tilde{E}^{\text{univ}})) \\ &= q^i (c_i((\tilde{u})^* \tilde{E}^{\text{univ}})) \\ &= q^i c_i(E^{\text{univ}}). \end{aligned}$$

On the other hand we obtain also

$$\begin{aligned} (\overline{F}_X \times \text{id}_{\tilde{\mathcal{L}}})^*(\text{id}_{\tilde{\mathcal{L}}} \times \overline{F}_{\mathcal{L}})^*(c_i(E^{\text{univ}})) &= (\text{id}_{\tilde{\mathcal{L}}} \times F_{\tilde{\mathcal{L}}})^*(\overline{F}_X \times \text{id}_{\tilde{\mathcal{L}}})^*(c_i(E^{\text{univ}})) \\ &= (\text{id}_{\tilde{\mathcal{L}}} \times F_{\tilde{\mathcal{L}}})^*(\overline{F}_X \times \text{id}_{\tilde{\mathcal{L}}})^*(1 \otimes c_i + \sum_{j=1}^{2g} \alpha_j \otimes a_i^{(j)} + [X] \otimes b_{i-1}) \\ &= (\text{id}_{\tilde{\mathcal{L}}} \times F_{\tilde{\mathcal{L}}})^*(1 \otimes c_i + \sum_{j=1}^{2g} \lambda_j \alpha_j \otimes a_i^{(j)} + q[X] \otimes b_{i-1}) \end{aligned}$$

This implies immediately the following fundamental proposition

**Proposition 2.5.** *The action of the geometric Frobenius  $\overline{F}_{\mathcal{L}}$  on the generating classes of the cohomology  $H^*(\tilde{\mathcal{L}}; \overline{\mathbb{Q}}_l)$  of the stack  $\mathcal{L}$  of rank  $n$  vector bundles on  $X$  is given as*

$$\begin{aligned}\overline{F}_{\mathcal{L}}^*(c_i) &= q^i c_i, \quad (i \geq 2) \\ \overline{F}_{\mathcal{L}}^*(b_i) &= q^{i-1} b_i, \quad (i \geq 1) \\ \overline{F}_{\mathcal{L}}^*(a_i^{(j)}) &= \lambda_j^{-1} q^i a_i^{(j)} \quad (i \geq 2, j = 1, \dots, 2g).\end{aligned}$$

**Corollary 2.6.** *The action of the geometric Frobenius  $F_{\tilde{\mathcal{L}}}$  on the cohomology of the open substack  $\tilde{\mathcal{L}}_{\text{semistable}}$  of semistable bundles is semisimple.*

**Proof.** The restriction morphism  $H^*(\tilde{\mathcal{L}}; \overline{\mathbb{Q}}_l) \rightarrow H^*(\tilde{\mathcal{L}}_{\text{semistable}}; \overline{\mathbb{Q}}_l)$  is surjective. As the cohomology  $H^*(\tilde{\mathcal{L}}; \overline{\mathbb{Q}}_l)$  is a semisimple  $\overline{\mathbb{Q}}_l[F_{\tilde{\mathcal{L}}}]$ -module, the same is also true for  $H^*(\tilde{\mathcal{L}}_{\text{semistable}}; \overline{\mathbb{Q}}_l)$ .  $\square$

**Remark.** The semisimplicity of the Frobenius here follows basically from the semisimplicity of the action of the Frobenius on the cohomology of a curve, using [18], p. 203. There the result is proven for abelian varieties over finite fields. As the first cohomology of a curve can be identified with that of its Jacobian, the desired result for curves follows immediately.

We have considered above only the case of bundles with trivial determinant. The same technique can be applied however for the stack of bundles of rank  $n$  and degree  $d$ , where  $\gcd(n, d) = 1$ , i.e.  $n$  and  $d$  are coprime numbers.

**Proposition 2.7.** *Suppose  $\gcd(n, d) = 1$  as above. Then the action of the Galois group  $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$  resp. the arithmetic Frobenius  $\psi$  on the cohomology  $H^*(M(n, d); \overline{\mathbb{Q}}_l)$  of the coarse moduli space  $M(n, d)$  of stable vector bundles of rank  $n$  and degree  $d$  is semisimple.*

**Proof.** Because  $\gcd(n, d) = 1$ , we have  $\tilde{\mathcal{L}}(n; d)_{\text{semistable}} = \tilde{\mathcal{L}}(n, d)_{\text{stable}}$ . Furthermore there is a canonical morphism

$$\tilde{\mathcal{L}}(n; d)_{\text{stable}} \rightarrow M(n, d),$$

given functorially for any scheme  $S/\mathbb{F}_q$  by the obvious functor

$$\mathcal{L}(n, d)_{\text{stable}}(S) \rightarrow M(n, d)(S),$$

where  $M(n, d)(S)$  denotes the discrete category of isomorphism classes of vector bundles of rank  $n$  and degree  $d$  (pointwise in  $S$ ) on  $X \times_{\mathbb{F}_q} S$ .

The Leray spectral sequence for this morphism is degenerating, because we have an equality of Poincaré polynomials

$$P(\tilde{\mathcal{L}}(n; d)_{\text{stable}}; t) = P(\overline{M}(n, d); t) \cdot P(B\mathbb{G}_m; t),$$

where the morphism  $\tilde{\mathcal{L}}_{\text{stable}} \rightarrow \overline{M}(n, d)$  is a fibration with fibre the classifying stack  $B\mathbb{G}_m$ .

Therefore  $H^*(\overline{M}(n, d); \overline{\mathbb{Q}}_l)$  is a submodule of  $H^*(\tilde{\mathcal{L}}_{\text{stable}}; \overline{\mathbb{Q}}_l)$ . As the last one is a semisimple module over  $\overline{\mathbb{Q}}_l[F_{\mathcal{L}}]$ ,  $H^*(\overline{M}(n, d); \overline{\mathbb{Q}}_l)$  is a semisimple module

over  $\overline{\mathbb{Q}}_l[\overline{F}_{M(n;d)}]$ , or equivalently the Frobenius  $\psi$  is acting as a semisimple endomorphism.  $\square$

**Proposition 2.8.** *Identifying the algebraic closure  $\overline{\mathbb{Q}}_l$  with the field  $\mathbb{C}$  of complex numbers, the expression for the formal trace*

$$\mathrm{tr}(\varphi^r \times \psi^s; H^*(\bar{\mathcal{L}}; \overline{\mathbb{Q}}_l)) = \sum_{i \geq 0} (-1)^i \mathrm{tr}(\varphi^r \times \psi^s; H^i(\bar{\mathcal{L}}; \overline{\mathbb{Q}}_l))$$

is absolutely convergent for  $s > r$ .

**Proof.** This is just an exercise in summing up geometric series. The formal series to be considered is given as

$$\left( \prod_{i=2}^n \left( \sum_{n=0}^{\infty} q^{-(si)} \right) \right) \cdot \left( \prod_{k=1}^{n-1} \sum_{m=0}^{\infty} q^{m(r-ks)} \right) \cdot \left( \prod_{j=1}^{2g} \prod_{i=1}^n (1 + |\lambda_j|^{r+s} q^{-is}) \right)$$

which implies the statement.  $\square$

**Remark.** The question is now of course, if there is a kind of Lefschetz trace formula in this general situation.

This is true at least in the case  $r = 0$ ,  $s = 1$ , which was considered in [3], [12]. The following trace formula holds

$$q^{\dim(\mathcal{L})} \sum_{i \geq 0} (-1)^i \mathrm{tr}(\psi; H^i(\bar{\mathcal{L}}; \overline{\mathbb{Q}}_l)) = \sum_{[E] \in \mathcal{L}(k)} \frac{1}{|\mathrm{Aut}^{(0)}(E)|},$$

where

$$\mathrm{Aut}^{(0)}(E) := \{\alpha \in \mathrm{Aut}(E)(k) \mid \det(\alpha) = 1\}$$

and  $\dim(\mathcal{L}) = n^2(g-1) + 1$ .

But as mentioned already in the introduction, the situation in the cases  $r = 0$  resp.  $r > 0$  is rather different, because the proof in the case  $r = 0$  works by studying  $\bar{\mathcal{L}}$  and the traces above using the filtration by the  $\bar{\mathcal{L}}_{\leq p}$  as done earlier. On the other hand the morphism  $\varphi : \bar{\mathcal{L}} \rightarrow \bar{\mathcal{L}}$  does not respect these open substacks.

### 3. SOME COMPLEMENTS AND EXAMPLES

#### Part 1: Vector bundles of rank 2 on the projective line

It seems worthwhile to study in detail the easiest example, namely the case of the projective line  $X = \mathbb{P}^1$  and vector bundles of rank  $n = 2$  with trivial determinant. We specialize our computations from section 2 to this case. By Proposition 2.3, we have in this special situation

$$H^*(\bar{\mathcal{L}}(\mathbb{P}^1; 2); \overline{\mathbb{Q}}_l) \cong \overline{\mathbb{Q}}_l[c_2, b_1],$$

where

$$b_1 \in H^2(\bar{\mathcal{L}}; \overline{\mathbb{Q}}_l), \quad c_2 \in H^4(\bar{\mathcal{L}}; \overline{\mathbb{Q}}_l)$$

For the formal trace we obtain

$$\begin{aligned} \mathrm{tr}(\varphi^r \times \psi^s; H^*(\bar{\mathcal{L}}(\mathbb{P}^1; 2); \overline{\mathbb{Q}}_l)) &= \sum_{i \geq 0} (-1)^i \mathrm{tr}(\varphi^r \times \psi^s; H^i(\bar{\mathcal{L}}(\mathbb{P}^1; 2); \overline{\mathbb{Q}}_l)) \\ &= \left( \sum_{m=0}^{\infty} q^{-2sm} \right) \left( \sum_{m=0}^{\infty} q^{(r-s)m} \right), \end{aligned}$$

which is convergent for  $r < s$ . We obtain therefore for  $r < s$

$$\mathrm{tr}(\varphi^r \times \psi^s; H^*(\bar{\mathcal{L}}(\mathbb{P}^1; 2); \overline{\mathbb{Q}}_l)) = (1 - q^{-2s})^{-1} (1 - q^{r-s})^{-1}$$

**Remark.** In this special case the fixed points under the action of  $(\varphi^r \times \psi^s)$  can be computed directly. The naive fixed point set can be computed as

$$(\bar{\mathcal{L}})^{\varphi^r \times \psi^s} = (\bar{\mathcal{L}}_{\text{semistable}})^{\varphi^r \times \psi^s} = (BSL(2)/\mathbb{F}_q \times \overline{\mathbb{F}}_q)^{\varphi^r \times \psi^s}$$

Here we have used the isomorphism of stacks

$$BSL(2)/\mathbb{F}_q \rightarrow \mathcal{L}(\mathbb{P}^1; 2)_{\text{semistable}},$$

given functorially for the categories of  $S$ -valued points,  $S/\mathbb{F}_q$  a scheme, by

$$E \mapsto \mathrm{pr}_2^*(E),$$

where  $E$  is a vector bundle of rank two over  $S$  with trivial determinant and  $\mathrm{pr}_2^*(E)$  is the pullback to  $\mathbb{P}^1 \times_{\mathbb{F}_q} S$  via the projection

$$\mathrm{pr}_2 : \mathbb{P}^1 \times_{\mathbb{F}_q} S \rightarrow S$$

This implies immediately, that  $\varphi$  acts as the identity on  $\mathcal{L}_{\text{semistable}}$ . But then we obtain

$$(\bar{\mathcal{L}})^{\varphi^r \times \psi^s} = (BSL(2)/\mathbb{F}_q \times \overline{\mathbb{F}}_q)^{\psi^s}$$

This does not depend on  $r$  at all, which obviously does not fit with the formula obtained above.

## Part 2: Affine Grassmannians and gauge groups

In this second part of section 3 we give a description of the moduli stack of vector bundles as a double quotient similar to the well known adelic description of vector bundles on a curve. These considerations are useful for our goal to make sense of a general trace formula, in order to do our computations of section 2 for an appropriate compactification of the stack  $\bar{\mathcal{L}}$  or  $\mathcal{L}$ .

For the following we consider a smooth projective curve  $X$  over  $\mathbb{F}_q$  with a closed  $\mathbb{F}_q$ -rational point  $\infty \in X$  (whose existence we assume for simplicity).  $\mathcal{O}_{X, \infty}$  is the local ring of  $X$  at  $\infty$ ,  $\mathcal{O}_{\infty}$  its completion,  $F_{\infty}$  the completion of the function field  $F = \mathbb{F}_q(X)$  at  $\infty$ .  $X^{(\infty)} = X \setminus \{\infty\}$  is an open subscheme of  $X$ .

Let  $\tilde{\mathcal{L}}(X; n)$  denote the algebraic stack of vector bundles of rank  $n$  on  $X$ , similarly  $\tilde{\mathcal{L}}(X^{(\infty)}; n)$  denotes the stack of vector bundles of rank  $n$  on  $X^{(\infty)}$ .

Finally, let  $\mathcal{L}(X; n)$  resp.  $\mathcal{L}(X^{(\infty)}; n)$  denote the corresponding stacks with trivialized determinant.

There is an obvious morphism  $r$  of stacks

$$r : \mathcal{L}(X; n) \rightarrow \mathcal{L}(X^{(\infty)}; n)$$

given by the restriction functor

$$E \in \mathcal{L}(X; n)(S) \mapsto (E \mid X^{(\infty)} \times_{\mathbb{F}_q} S) \in \mathcal{L}(X^{(\infty)}; n)(S),$$

where  $S/\mathbb{F}_q$  is any scheme over  $\mathbb{F}_q$ .

The fibre over a closed point  $(E^{(0)}; \delta^0) \in \mathcal{L}(X^{(\infty)}; n)$  can be described as the stack, given functorially by  $S/\mathbb{F}_q \mapsto \underline{\text{Gr}}(E^{(0)})(S)$ , where  $\underline{\text{Gr}}(E^{(0)})(S)$  denotes the category of triples  $((E, \delta), n)$ , where  $(E, \delta) \in \mathcal{L}(X; n)(S)$  that is,  $E$  is a rank  $n$  vector bundle over  $X \times_{\mathbb{F}_q} S$ ,  $\delta : \det(E) \xrightarrow{\sim} \mathcal{O}_{X \times_{\mathbb{F}_q} S}$  a trivialisation of the associated determinant bundle and  $u$  is an isomorphism of vector bundles

$$u : E \mid_{X^{(\infty)} \times_{\mathbb{F}_q} S} \xrightarrow{\sim} \text{pr}_2^*(E^{(0)}),$$

where  $\text{pr}_2 : X^{(\infty)} \times_{\mathbb{F}_q} S \rightarrow S$  is the projection on the second factor. Additionally, the induced determinantal isomorphism  $\det(u)$  has to make the obvious diagram of determinant bundles and trivialisations commutative.

**Remark.** For simplicity we have written  $\underline{\text{Gr}}(E^{(0)})$  instead of  $\underline{\text{Gr}}((E^{(0)}; \delta^{(0)}))$ .

As is known (see for example [10]), there is a purely local description of  $\underline{\text{Gr}}(E^{(0)})$  as follows:

Consider the group-valued functors

$$\begin{aligned} \underline{\text{affine schemes}}/\mathbb{F}_q &\longrightarrow \underline{\text{groups}} \\ S &\longmapsto \text{SL}(n; \mathcal{O}_S[[t_\infty]]), \\ \text{resp. } S &\longmapsto \text{SL}(n; \mathcal{O}_S((t_\infty))) \end{aligned}$$

Then the first functor is represented by an algebraic group (not of finite type) and the second functor is represented by an ind-algebraic group. We denote these two objects somewhat informally as  $\underline{\text{SL}}(n; \mathcal{O}_\infty)$  resp.  $\underline{\text{SL}}(n; F_\infty)$ .

The quotient

$$\underline{\text{SL}}(n; F_\infty)/\underline{\text{SL}}(n; \mathcal{O}_\infty)$$

exists as a stack (or even an ind-algebraic scheme) and is isomorphic to the stack  $\underline{\text{Gr}}(E^{(0)})$ . (see [10] and [9].)

Choosing a pair  $(E^{(0)}, \delta^{(0)}) \in \mathcal{L}(X^{(\infty)}; n)(\mathbb{F}_q)$ , for example the trivial bundle, we have the following diagram of stacks

$$\underline{\text{Gr}}(E^{(0)}) \rightarrow \mathcal{L}(X; n) \rightarrow \mathcal{L}(X^{(\infty)}; n),$$

where  $\underline{\text{Gr}}(E^{(0)}) \rightarrow \mathcal{L}(X; n)$  is the morphism of stacks, given functorially by

$$((E, \delta), u) \mapsto (E, \delta)$$

using the notations from above.



Now we indicate the computation of the cohomologies in this case.

**Proposition 3.1.** *The filtration of the stack  $\mathcal{L}(X; n)$  by the open substacks  $\mathcal{L}(X; n)_{\leq p}$  induces a corresponding filtration  $\underline{\underline{\text{Gr}}}(E^{(0)})_{\leq p}$  of  $\underline{\underline{\text{Gr}}}(E^{(0)})$ .*

This can be used to compute the cohomology of the infinite Grassmannian  $\underline{\underline{\text{Gr}}}(E^{(0)})$  as follows:

**Proposition 3.2.** *The cohomology ring  $H^*(\underline{\underline{\text{Gr}}}(E^{(0)}) \times \overline{\mathbb{F}}_q; \overline{\mathbb{Q}}_l)$  is given as  $\overline{\mathbb{Q}}_l[b_1, \dots, b_{n-1}]$  where the  $b_i$  ( $i = 1, \dots, n-1$ ) are the restrictions of the corresponding classes  $b_i$  in the cohomology ring  $H^*(\bar{\mathcal{L}}; \overline{\mathbb{Q}}_l)$ .*

**Proof.** We can consider the universal object

$$(E^{\text{univ}}, \delta^{\text{univ}}), u : E^{\text{univ}}|_{X^{(\infty)} \times_{\mathbb{F}_q} \underline{\underline{\text{Gr}}}(E^{(0)})} \xrightarrow{\sim} \text{pr}_2^*(\mathcal{O}_{X^{(\infty)}}^n)$$

over  $X \times_{\mathbb{F}_q} \underline{\underline{\text{Gr}}}(E^{(0)})$ . We can write down (for example by pullback) the Chern classes of  $E^{\text{univ}}$  as

$$c_i(E^{\text{univ}}) = 1 \otimes c_i + \sum_{j=1}^{2g} \alpha_j \otimes a_i^{(j)} + [X] \otimes b_{i-1}$$

for  $i \geq 1$  in  $H^{2i}((X \times_{\mathbb{F}_q} \underline{\underline{\text{Gr}}}(E^{(0)})) \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q; \overline{\mathbb{Q}}_l)$ .

Here the  $c_i, b_i, a_i^{(j)}$  are just the restrictions of the corresponding classes under the morphism  $\underline{\underline{\text{Gr}}}(E^{(0)}) \rightarrow \mathcal{L}$  considered above. On the other hand we have the identity

$$c_i(E^{\text{univ}})|_{(X^{(\infty)} \times \underline{\underline{\text{Gr}}}(E^{(0)})) \times \overline{\mathbb{F}}_q} = \varphi^*(\text{pr}_1^*(c_i(E^{(0)})))$$

by the very definition of  $(E^{\text{univ}}, \varphi^{\text{univ}})$ .

But  $c_i(E^{(0)}) = 0$  for  $i \geq 1$ , therefore we obtain for the restrictions

$$\begin{aligned} c_i|_{\underline{\underline{\text{Gr}}}(E^0)} &= 0 & (i \geq 1) \\ a_i^{(j)}|_{\underline{\underline{\text{Gr}}}(E^0)} &= 0 & (i \geq 1, j = 1, \dots, 2g) \end{aligned}$$

It remains to show that  $b_1, \dots, b_{n-1}$  are generating elements of the cohomology and are polynomially independent. This can be done using the stratification mentioned above in a similar treatment as in the corresponding statements for the cohomology of the stack  $\bar{\mathcal{L}}$ .  $\square$

**Remark.** In a similar way, using a yet to be defined version of  $l$ -adic cohomology appropriate for the stack  $\mathcal{L}(X^{(\infty)} \times \overline{\mathbb{F}}_q; n)$ , we could consider the morphism  $r : \mathcal{L}(X; n) \rightarrow \mathcal{L}(X^{(\infty)}; n)$ , given as restriction of bundles to  $X^{(\infty)}$ . For the universal bundles we have the pullback situation

$$r^*(E^{\text{univ}}(\text{over } X^{(\infty)} \times \mathcal{L}(X^{(\infty)}; n))) = E^{\text{univ}}(\text{over } X \times \mathcal{L}(X; n))$$

Again for  $E^{\text{univ}}$  over  $X^{(\infty)} \times \mathcal{L}(X^{(\infty)}; n)$ , we could consider the Chern classes and then obtain the following description

$$c_i(E^{\text{univ}}) = 1 \otimes c_i + \sum_{j=1}^{2g} \alpha_j \otimes a_j^{(i)}, \quad (1 \leq i \leq n)$$

because here we have  $[X]|H^2(\bar{X}_\infty) = 0$ .

Then the cohomology  $H^*(\mathcal{L}(X^{(\infty)}; n) \times \bar{\mathbb{F}}_q; \bar{\mathbb{Q}}_l)$  is given as

$$\bar{\mathbb{Q}}_l[c_1, \dots, c_n] \otimes \bigotimes_{i=1}^n \Lambda[a_i^{(1)}, \dots, a_i^{(2g)}]$$

in the sense of Proposition 2.3 and the Leray spectral sequence for the morphism of stacks

$$\bar{r} : \bar{\mathcal{L}}(X; n) \rightarrow \bar{\mathcal{L}}(X^{(\infty)}; n)$$

would degenerate.

We would get this description along the following lines. As the cohomology classes

$$\begin{aligned} &\{c_i(\bar{\mathcal{L}}(X)), a_i^{(j)}(\bar{\mathcal{L}}(X)) \mid i = 1, \dots, n, j = 1, \dots, 2g\} \\ &\{b_i(\bar{\mathcal{L}}(X)) \mid i = 1, \dots, n-1\} \end{aligned}$$

are independent in the sense of Proposition 2.3, the same is also true for their preimages

$$\{c_i(\bar{\mathcal{L}}(X^{(\infty)})); a_i^{(j)}(\bar{\mathcal{L}}(X^{(\infty)})) \mid i = 1, \dots, n, \text{ and } j = 1, \dots, 2g\}.$$

On the other hand comparing the Poincaré polynomials we would get

$$P(\mathcal{L}(X); t) = P(\underline{\text{Gr}}(E^{(0)}; t)P(\mathcal{L}(X^{(\infty)}); t)$$

From this and the corresponding computations of  $H^*(\bar{\mathcal{L}}(X); \bar{\mathbb{Q}}_l)$  and the cohomology  $H^*(\underline{\text{Gr}}(E^{(0)}); \bar{\mathbb{Q}}_l)$  we would then obtain the desired description of the cohomology rings and the Leray spectral sequence.

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